Large N Expansion. Vector Models *

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Large-N expansion of quantum field theory (QFT) models with internal $U(N)$ (or $O(N)$) "flavor" symmetry, where the fundamental matter fields belong to the vector representation (*vector QFT models* for short), is one of the principal, most well understood and systematically developed non-perturbative methods. Here the term "non-perturbative" means that large-N expansion is qualitatively different from standard ("naive") perturbation theory w.r.t. coupling constant(s), as its diagrams involve new types of internal propagator lines corresponding to auxiliary "flavor"-singlet composite fields which are given by infinite resummation of (subsets of) ordinary Feynman diagrams. The latter results in improved ultraviolet (UV) behavior of large-N diagrams which coupled with the fact that $1/N$ is dimensionless expansion parameter makes renormalizable those vector QFT models which are non-renormalizable w.r.t. the ordinary perturbation theory. Another important general property of large-N expansion is that it exhibits *linear* realization of $U(N)$ (or $O(N)$) "flavor" symmetry in QFT models with a *nonlinearly* realized "flavor" symmetry, *i.e.*, the nonlinear sigma-models.

Large-N expansion is the main instrument in uncovering and for explicit description of the following important properties of QFT (henceforth dimensionality of space-time will be denoted by D):

(i) $D = 2$ QFT: dynamical breakdown of classical conformal symmetry via dimensional transmutation of coupling constants together with asymptotic freedom, as well as construction of higher local quantum conserved currents in $D = 2$ integrable models.

(ii) $D \geq 3$ QFT: dynamical breaking of continuous and discrete (space- and time-reflection) symmetries; non-trivial phase structure (several distinct types of phases with multiple order parameters) and the pertinent phase transitions; non-perturbative particle spectra qualitatively different in the various phases; dynamical mass generation for the fundamental N-component matter fields; dynamical generation of massive gauge bosons where the standard Higgs mechanism is inoperative; particle confinement in some of the phases, explicit appearance of composite bosons and fermions; renormalization of non-renormalizable (w.r.t. ordinary perturbation theory) QFT models.

(iii) Further applications of large-N expansion of vector models in various areas of QFT and statistical mechanics (*i.e.*, Euclidean QFT) include: finite-size effects (finite-size scaling in the nonlinear sigma-models); stochastic Langevin equation in dissipative dynamics; finite-temperature QFT (dimensional reduction crossover at high temperature); Bose-Einstein condensation in weakly interacting Bose gas; multicritical points and double scaling limit; for a comprehensive review, see ref.[1].

Derivation of large-N expansion via functional integral techniques is based on the following general prescription: (a) Introduce appropriate set of auxiliary "flavor"-singlet fields and rewrite the original action in a (classically) equivalent form which is quadratic w.r.t. fundamental N-component matter fields; (b) In the functional integral expression for the generating functional of the quantum correlation functions perform the Gaussian functional integral over the N-component matter fields to obtain an effective action depending only on "flavor"-singlet fields, where the factor N appears as a common factor in front of it in the same way as the Planck constant appears as a common factor $1/\hbar$ in front of the ordinary classical action in the standard functional integral; (c) Then the large-N expansion becomes "semiclassical" expansion around saddle points of the effective "flavor"-singlet action, which can be viewed as vacuum expectation values of the pertinent "flavor"-singlet fields in the leading order w.r.t. $1/N$ (cf. Eqs.(11) below).

Our first example will be the large-N expansion in $D = 2$ $O(N)$ nonlinear sigma-model whose Lagrangian

*Preliminary version of a contribution to the "Quantum Field Theory. Non-Perturbative QFT" topical area of *Modern Encyclopedia of Mathematical Physics (SELECTA)*, eds. Aref'eva I, and Sternheimer D, Springer (2007) is given by:

$$\mathcal{L}_{NLSM} = \partial_+\varphi^a\partial_-\varphi_a, \qquad \vec{\varphi}^{\,2} = N/g \quad , \quad \partial_\pm = \frac{1}{2}\left(\frac{\partial}{\partial x^0} \pm \frac{\partial}{\partial x^1}\right) \qquad \vec{\varphi} = (\varphi^1, \ldots \varphi^N) \tag{1}$$

The large-N expansion is obtained from the generating functional of the time-ordered correlation functions:

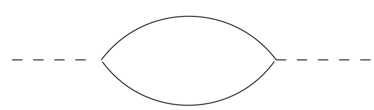
$$Z[J] = \int \mathcal{D}\vec{\varphi} \prod_x \delta\left(\vec{\varphi}^{\,2} - N/g\right) \exp\left\{i\int d^2x \left[\partial_+\vec{\varphi}\,\partial_-\vec{\varphi} + \left(\vec{J},\vec{\varphi}\right)\right]\right\} \tag{2}$$

$$= \int \mathcal{D}\vec{\varphi}\,\mathcal{D}\alpha \exp\left\{i\int d^2x \left[\partial_+\vec{\varphi}\,\partial_-\vec{\varphi} - \frac{1}{2}\alpha\left(\vec{\varphi}^{\,2} - N/g\right) + \left(\vec{J},\vec{\varphi}\right)\right]\right\}$$

$$= \int \mathcal{D}\alpha \exp\left\{-\frac{N}{2}S_1[\alpha] + \frac{i}{2}\int d^2x\, d^2y \left(\vec{J}(x), (-\partial^2+\alpha)^{-1}\vec{J}(y)\right)\right\}$$

$$S_1[\alpha] \equiv \mathrm{Tr}\ln(-\partial^2+\alpha) - \frac{i}{g}\int d^2x\,\alpha \quad , \quad \partial^2 = -\left(\frac{\partial}{\partial x^0}\right)^2 + \left(\frac{\partial}{\partial x^1}\right)^2 \tag{3}$$

by expanding the effective α-field action (3) around its constant saddle point $\widehat{\alpha} \equiv m^2$, *i.e.*, $\alpha(x) = m^2 + \frac{1}{\sqrt{N}}\tilde{\alpha}(x)$. From the stationary equation $\delta S_1[\alpha]/\delta\alpha \mid_{\alpha=m^2} = 0$ one obtains $m^2 = \hat{\mu}^2 e^{-4\pi/g}$, where $\hat{\mu}$ is a renormalization mass scale appearing due to renormalization of the UV divergence coming from the first term in (3) (see Eq.(13) below). Thus, the "Goldstone" fields $\vec{\varphi}$ acquire dynamically generated mass (squared) $\widehat{\alpha} \equiv m^2$, classical conformal invariance of (2) is broken due to the dimensional transmutation (the dimensionless coupling g is replaced by m^2), and the classically nonlinearly realized $O(N)$ "flavor" symmetry becomes linearly realized on the quantum level. From (3) one arrives at the large-N diagram technique with (free) propagators in momentum space: $\left\langle\varphi^a\,\varphi^b\right\rangle_{(0)} = -i\left(m^2+p^2\right)^{-1}\delta^{ab}$, $\langle\tilde{\alpha}\,\tilde{\alpha}\rangle_{(0)} = \left(\Sigma\left(p^2\right)\right)^{-1}$ with $\Sigma\left(p^2\right) = \int\frac{d^2k}{(2\pi)^2}\left[\left(m^2+k^2\right)\left(m^2+(p-k)^2\right)\right]^{-1}$, and tri-linear $\tilde{\alpha}\varphi\varphi$-vertices, where one-loop φ-tadpoles and subdiagrams of the form in the picture below are forbidden (solid lines depict φ propagators, dashed lines depict $\tilde{\alpha}$ propagators).



The diagrams of the large-N expansion of vector QFT models still contain UV divergences which can be systematically renormalized both in $D = 2$ and $D \geq 3$ by a version of the mass-independent ("soft-mass") momentum-space subtraction procedure of Zimmermann-Lowenstein, which in turn is based on BPHZ (Bogoliubov-Parasiuk-Hepp-Zimmermann) renormalization scheme. The mass-independent momentum-space subtraction renormalization (for details, see the link BPHZL Renormalization) has the advantage over other renormalization schemes that it can be applied simultaneously and in an uniform way in all phases of the pertinent QFT models, especially in those of them with phases containing massless particles where particular care is needed to avoid possible infra-red singularities.

A remarkable property of the large-N expansion in nonlinear sigma models is that the nonlinearity of the "Goldstone" field $\vec{\varphi}(x)$ is preserved on the quantum level as an identity on the correlation functions, in spite of the manifest linear $O(N)$ symmetry of the large-N diagrams:

$$\left\langle\mathcal{N}\left[\vec{\varphi}^2 P(\vec{\varphi},\partial\vec{\varphi})\right](x)\ldots\right\rangle = \mathrm{const}\left\langle\mathcal{N}\left[P(\vec{\varphi},\partial\vec{\varphi})\right](x)\ldots\right\rangle \tag{4}$$

where $P(\vec{\varphi},\partial\vec{\varphi})$ is arbitrary local polynomial of the fundamental fields and their derivatives, and $\mathcal{N}[\ldots]$ indicates BPHZL-renormalized normal product of the corresponding composite fields.

Using the *BPHZL*-renormalized large-N expansion one can explicitly construct the higher quantum local conserved currents $\mathcal{J}_\pm^{(s)}$ for the model (1) ($\partial_+\mathcal{J}_-^{(s)} + \partial_-\mathcal{J}_+^{(s)} = 0$, $s = 3, 5, \ldots$, where s indicates the $D = 2$ Lorentz spin of the corresponding higher conserved charge). Their existence is of profound importance as they imply quantum integrability of the $O(N)$ nonlinear sigma-model (1). The first non-trivial higher quantum local conserved current is of the form:

$$\mathcal{J}_-^{(3)} = \mathcal{N}\left[\left(\partial_-^2\vec{\varphi}\right)^2\right] + a_1\mathcal{N}\left[\left((\partial_-\vec{\varphi})^2\right)^2\right] \quad , \quad \mathcal{J}_+^{(3)} = \left(\frac{1}{2}+a_2\right)\mathcal{N}\left[(\partial_-\vec{\varphi})^2\,\alpha\right] + a_3\partial_-^2\alpha \tag{5}$$

where all coefficients $a_{1,2,3} = O(1/N)$ are expressed in terms of one-particle irreducible correlation functions and their derivatives in momentum space at zero external momenta. Their explicit form can be found order by order in $1/N$ from the renormalized large-N diagram technique described above. Let us stress, that (5) as well as its higher counterparts $\mathcal{J}_\pm^{(s)}$ for $s = 5, 7, \ldots$ do not have analogues in the classical conformally invariant $O(N)$ nonlinear sigma-model (1).

As a second non-trivial example let us consider the $D = 3$ gauged nonlinear sigma-models with fermions $(GNLSM + F)_3$, with internal symmetry $U(N)$("flavor") $\times\, U(n)$("color" gauge) $(n < N)$. Special physically relevant cases of the latter are the supersymmetric nonlinear sigma-models on complex projective and Grassmannian manifolds (*e.g.*, by taking $e^2 \to \infty$, $\varepsilon = 1$ in (6) below). On the other hand, $(GNLSM + F)_3$ themselves can be viewed as special fixed points of general $D = 3$ (non-Abelian) Higgs models with fermions, containing "non-renormalizable" four-fermion couplings. $(GNLSM + F)_3$ are of particular physical interest since their large-N expansion explicitly displays all the fundamental non-perturbative properties listed above under (ii). Below we will discuss for simplicity large-N expansion for the Abelian $(GNLSM + F)_3$ $(n = 1$, the non-Abelian case being a straightforward generalization).

The pertinent Lagrangian reads:

$$\mathcal{L}_{\mathrm{GNLSM+F}} = -\left(\nabla^\nu(A)\varphi_a\right)^*\left(\nabla_\nu(A)\varphi^a\right) + i\bar{\psi}_a\gamma^\nu\nabla_\nu^{(\varepsilon)}(A)\psi^a + \frac{g}{4N\mu}\left(\bar{\psi}_a\psi^a\right)^2 - \frac{N}{4e^2\mu}F_{\nu\lambda}(A)F^{\nu\lambda}(A) \tag{6}$$

with constraints $\varphi_a^*\varphi^a - N\mu/T = 0$, $\bar{\psi}_a\varphi^a = \varphi_a^*\psi^a = 0$. Here the following notations are used: $\nabla_\nu(A)\varphi^a = (\partial_\nu + iA_\nu)\,\varphi^a$, $\nabla_\nu^{(\varepsilon)}(A)\psi^a = (\partial_\nu + i\varepsilon A_\nu)\,\psi^a$, $F_{\nu\lambda}(A) = \partial_\nu A_\lambda - \partial_\lambda A_\nu$, where the "flavor" indices $a = 1, \ldots, N$ and the space-time indices $\nu, \lambda = 0, 1, 2$; ε is the ratio of fermionic to bosonic electric charges; γ_ν are the standard $D = 3$ Dirac gamma-matrices; μ denotes a common mass scale exhibiting the dimensionfull nature of the coupling constants $T/\mu, g/\mu, e^2\mu$. Note the presence of the "non-renormalizable" four-fermion (*Gross-Neveu*) term in (6).

Apart from the continuous $U(N)$("flavor") $\times\, U(1)$(gauge) symmetry, $(GNLSM + F)_3$ (6) is invariant also under the discrete space-time transformations – space (P-) and time (T-) reflections: $\varphi^{(P,T)}(x) = \eta_{P,T}\varphi(x_{P,T})$, $\psi^{(P,T)}(x) = \eta_{P,T}\gamma_{1,2}\varphi(x_{P,T})$, $A^{(P)}(x) = (A_0, -A_1, A_2)(x_{P,T})$, $A^{(T)}(x) = (A_0, -A_1, -A_2)(x_{P,T})$, where $x_P = (x^0, -x^1, x^2)$, $x_T = (-x^0, x^1, x^2)$ and $|\eta_{P,T}| = 1$. Note that fermionic mass term reverses sign under P, T-reflection: $\bar{\psi}^{(P,T)}\psi^{(P,T)}(x) = -\bar{\psi}\psi(x_{P,T})$ and due to its absence in (6) the classical $(GNLSM + F)_3$ is P, T-invariant. Therefore, P, T-reflection symmetries can be viewed as $D = 3$ analogues of the chiral symmetry in $D = 4$ gauge theories with massless chiral fermions.

Introducing a set of auxiliary $U(N)$-singlet fields (real scalar α,σ and complex fermionic ρ) one can rewrite the action (6) in the following (classically) equivalent form:

$$\begin{aligned} L_{\mathrm{GNLSM+F}} = &-\left(\nabla^\nu(A)\varphi_a\right)^*\left(\nabla_\nu(A)\varphi^a\right) - \alpha\left(\varphi_a^*\varphi^a - N\mu/T\right) + i\bar{\psi}_a\gamma^\nu\nabla_\nu^{(\varepsilon)}(A)\psi^a - \sigma\bar{\psi}_a\psi^a \\ &- \frac{N\mu}{g}\,\sigma^2 + \varphi^a\left(\bar{\psi}_a\rho\right) + \left(\bar{\rho}\psi^a\right)\varphi_a^* - \frac{N}{4e^2\mu}F_{\nu\lambda}(A)F^{\nu\lambda}(A) \end{aligned} \tag{7}$$

Derivation of the large-N expansion for the quantum generating functional $Z[J_\Phi]$ of (7) proceeds along similar lines as for the $D = 2$ $O(N)$ nonlinear sigma-model (1)–(3). Unlike the $D = 2$ case, in $D \geq 3$ the fundamental N-component scalar field may acquire non-zero vacuum expectation value for certain range of the parameters, therefore, it is appropriate to split it in two parts – parallel and orthogonal w.r.t. direction of the (possible) vacuum expectation value: $\vec{\varphi} = \vec{\varphi}_{||} + \vec{\varphi}_\perp$. Without loss of generality one may choose $\vec{\varphi}_{||} = (0, \ldots, 0, N^{\frac{1}{2}}\varphi_{||})$ and $\vec{\varphi}_\perp = (\varphi_1, \ldots, \varphi_{N-1}, 0)$. Then performing the Gaussian functional integration w.r.t. $\vec{\varphi}_\perp$, ψ one gets:

$$Z[J_\Phi] = \int \mathcal{D}\vec{\varphi}_\perp\mathcal{D}\psi\mathcal{D}\varphi_{||}\mathcal{D}\alpha\mathcal{D}\sigma\mathcal{D}\rho\mathcal{D}A_\mu\,\exp\Big\{i\int d^3x\Big[L_{\mathrm{GNLSM+F}} + \sum_{\Phi=\varphi,\psi,\ldots} J_\Phi(x)\Phi(x)\Big]\Big\} \tag{8}$$

$$= \int \mathcal{D}\varphi_{||}\mathcal{D}\alpha\mathcal{D}\sigma\mathcal{D}\rho\mathcal{D}A_\mu\,\exp\Big\{iNS_1\left[\varphi_{||}, \alpha, \sigma, \rho, A\right] + iS_2[J_\Phi]\Big\} \tag{9}$$

Here the effective action reads:

$$
S_1\left[\varphi_{||},\alpha,\sigma,\rho,A\right] = i(1-1/N)\,\mathrm{Tr}\ln\Delta_B - i\,\mathrm{Tr}\ln\Delta_F
$$
$$
+\int d^3x\Big[-\frac{1}{2}\varphi^*_{||}\Delta_B\varphi_{||} - \alpha\,\mu/T - \sigma^2\mu/g - \frac{1}{4e^2\mu}F_{\nu\lambda}(A)F^{\nu\lambda}(A)\Big] \tag{10}
$$

where $\Delta_F \equiv i\gamma^\lambda\nabla^{(\varepsilon)}_\nu(A)-\sigma$, $\Delta_B \equiv -\nabla^\nu(A)\nabla_\nu(A)+\alpha+\bar\rho\Delta_F^{-1}\rho$, and $S_2[J_\Phi]$ contains the terms with the sources.

Because of Lorentz invariance of the vacuum only $\varphi_{||}$, α and σ may have non-zero constant stationary values $\widehat{\varphi}_{||}\equiv v$, $\widehat{\alpha}\equiv m^2_\varphi$, $\widehat{\sigma}\equiv m_\psi$ where:

$$
\langle\varphi^a\rangle = N^{\frac{1}{2}}\left[v\delta^a_N + O(N^{-1})\right]\ ,\ \langle\alpha\rangle = m^2_\varphi + O(N^{-1})\ ,\ \left\langle\bar\psi\psi\right\rangle = \frac{2N\mu}{g}\langle\sigma\rangle = \frac{2N\mu}{g}\left[m_\psi + O(N^{-1})\right] \tag{11}
$$

Thus, the saddle-point equations acquire the form:

$$
\frac{\delta S_1}{\delta\varphi^*_{||}} = -m^2_\varphi\,v = 0\ ,\ \frac{\delta S_1}{\delta\alpha} = \frac{m_\varphi}{4\pi} - \left[|v|^2 + \mu\left(\frac{1}{T_c}-\frac{1}{T}\right)\right] = 0\ ,\ \frac{\delta S_1}{\delta\sigma} = -2m_\psi\left[\frac{m_\psi}{4\pi} - \mu\left(\frac{1}{T_c}-\frac{1}{g}\right)\right] = 0 \tag{12}
$$

The dimensionless constant $T_c = 4\pi\mu/\hat\mu$ arises in the evaluation of the UV-divergent integrals appearing in the variational derivatives of S_1 which are renormalized according to the "soft-mass" BPHZL subtraction scheme with arbitrary scale $\hat\mu$ (in particular, one may take $\hat\mu=\mu$) :

$$
i\frac{\delta\,\mathrm{Tr}\ln\Delta_B}{\delta\alpha}\Bigg|_{\widehat{\alpha}=m^2_\varphi,\dots,\widehat{\rho}=0} = \left\{i\int d^Dp/(2\pi)^D\left[m^2_\varphi+p^2\right]^{-1}\right\}^{\mathrm{ren}}
$$
$$
= i\int d^Dp/(2\pi)^D\left[\left(m^2_\varphi+p^2\right)^{-1} - \left(\hat\mu^2+p^2\right)^{-1}\right] = \begin{cases}\frac{1}{4\pi}\ln\left(m^2_\varphi/\hat\mu^2\right) & \text{for } D=2\\ \frac{1}{4\pi}\left(m_\varphi-\hat\mu\right) & \text{for } D=3\end{cases} \tag{13}
$$

and similarly for $-i\left\{\delta\,\mathrm{Tr}\ln\Delta_F/\delta\sigma\right\}\big|_{\widehat{\sigma}=m_\psi,A=0}$.

The solutions of the saddle-point equations (12) yield the following phase structure of $(GNLSM+F)_3$ (6) characterized by *two* order parameters $\langle\vec\varphi\rangle$, $\left\langle\bar\psi\psi\right\rangle$ (11):

(I) $U(N)$("flavor")$\times U(1)$(gauge) and P,T-symmetric "high-temperature" phase for $T>T_c$ and $0<g<T_c$, where: $v=0$, $m_\varphi = 4\pi\mu\,(1/T_c-1/T)$, $m_\psi=0$.

(II) $U(N)$("flavor")$\times U(1)$(gauge) symmetric "high-temperature" phase with spontaneous breakdown of discrete P,T-reflection symmetries due to dynamical generation of fermionic mass m_ψ for $T>T_c$ and either $g<0$ or $T_c<g<2T_c$, where: $v=0$, $m_\varphi=4\pi\mu\,(1/T_c-1/T)$, $m_\psi = 4\pi\mu\,(1/T_c-1/g)$.

(III) P,T-symmetric "low-temperature" phase with spontaneous breakdown of internal $U(N)$("flavor")$\times U(1)$(gauge) due to non-zero $\langle\vec\varphi\rangle$ (11) for $T<T_c$ and $0<g<T_c$, where: $|v|^2=\mu\,(1/T-1/T_c)$, $m_\varphi=0$, $m_\psi=0$.

(IV) "Low-temperature" phase with spontaneous breakdown of both the discrete P,T-symmetries (as in phase (II)) and internal symmetry (as in phase (III)) for $T<T_c$ and either $g<0$ or $T_c<g<2T_c$, where: $|v|^2=\mu\,(1/T-1/T_c)$, $m_\varphi=0$, $m_\psi=4\pi\mu\,(1/T_c-1/g)$.

Let us recall that P,T-reflection symmetries are $D=3$ analogues of the fermionic chiral symmetry in $D=4$.

The restriction $g<2T_c$ above originates from the stability requirement for the quantum effective potential of $(GNLSM{+}F)_3$ (6). According to the general definition it is given as a Legendre transform of the logarithm of the quantum generating functional (8): $\mathcal{U}\left(\langle\vec\varphi\rangle,\langle\bar\psi\psi\rangle\right) = -i\ln Z\left[J_\varphi, J_{\bar\psi\psi}\right] - \left(J^*_{\varphi\,a}\langle\varphi^a\rangle + \langle\varphi^*_a\rangle J^a_\varphi + J_{\bar\psi\psi}\langle\bar\psi\psi\rangle\right)$. In the large-$N$ limit one obtains (cf. the relations (11)): $N^{-1}\mathcal{U}\left(\langle\vec\varphi\rangle,\langle\bar\psi\psi\rangle\right) = \mathcal{U}_1\left(\langle\vec\varphi\rangle,\langle\sigma\rangle\right) - g/4\mu\,\left(\delta\mathcal{U}_1/\delta\langle\sigma\rangle\right)^2$ where $\mathcal{U}_1\left(\langle\vec\varphi\rangle,\langle\sigma\rangle\right) = 1/6\pi\,\left(|\langle\sigma\rangle|^3 - \langle\alpha\rangle^{3/2}\right) - \mu|\langle\sigma\rangle|^2\,(1/T_c-1/g) + \langle\sigma\rangle\left[|\langle\vec\varphi\rangle|^2 + \mu\,(1/T_c-1/T)\right]$.

All transitions between any pair of the above phases are second-order on the lines $T=T_c$ and $g=T_c$ on the (T,g) parameter plane. On the other hand, the line $g=0$ corresponds to first-order phase transitions between phases (I) and (II) for $T>T_c$, and between phases (III) and (IV) for $T<T_c$.

All four phases exhibit qualitatively different non-perturbative particle spectra. The spectra are directly derived from the momentum-space pole structure of the propagators in the pertinent large-N diagrams, where the propagators themselves are determined from the quadratic part of the expansion of the large-N effective action (10) around its saddle points. The highlights of these spectra include appearance of composite bosons and fermions in phases (II) and (IV), "confinement" of part of the fundamental N-component matter fields (φ, ψ) in phases (III) and (IV). The most interesting effect occurs in phase (II), which contains massive gauge bosons due to dynamical generation in the large-N effective action (10) of a P, T-noninvariant topological Chern-Simmons term $1/16\pi \int d^3x\, \varepsilon^{\kappa\lambda\nu} A_\kappa F_{\lambda\nu}(A)$. In all other phases the gauge bosons are "confined" due to appearance of $\sqrt{p^2}$-singularity in the A_ν-propagators. Thus, in spite of the unbroken gauge symmetry in phases (I) and (II), massless gauge bosons are absent there. Also, the standard Higgs mechanism for generating masses of gauge bosons does not operate in phases (III) and (IV) in spite of the spontaneous breakdown of the gauge symmetry there.

It is also worth mentioning that at the critical point $T = T_c$, $g = T_c$ and upon taking the scaling limit $(GNLSM + F)_3$ (6) becomes the $D = 3$ supersymmetric non-linear sigma-model on the complex projective space CP^{N-1}: $\mathcal{L}_{\mathrm{susy}CP^{N-1}} = -\left(\nabla^\nu(A)\varphi_a\right)^* \left(\nabla_\nu(A)\varphi^a\right) + i\bar{\psi}_a\gamma^\nu\nabla_\nu(A)\psi^a + \frac{T_c}{4N\mu}\left(\bar{\psi}_a\psi^a\right)^2$ with constraints $\varphi^*_a\varphi^a - N\mu/T_c = 0 \quad , \quad \bar{\psi}_a\varphi^a = \varphi^*_a\psi^a = 0$. This is a *non-trivial* $D = 3$ conformal field theory with a well-defined renormalizable large-N expansion where all relevant anomalous conformal dimensions (some of them describing the critical behaviour of $(GNLSM + F)_3$ (6) in the vicinity of the second-order phase transitions) can be explicitly computed order by order in $1/N$ from the large-N diagram techniques.

Further Reading.

Ref.[1] contains a comprehensive review (together with an extensive list of references) of most of the relevant aspects and applications of large-N expansion of vector QFT models, especially those mentioned under (iii) above (see also the book [2]). More details about application of large-N expansion to construct higher local quantum conserved currents in $D = 2$ integrable QFT models with $O(N)$ (or $U(N)$) internal symmetry can be found in refs.[3,4] and references therein. For a systematic renormalization of the large-N expansion, including proofs of renormalizability of QFT models which are non-renormalizable within the standard perturbation theory, see refs.[5,6,7] and references therein. Further details about application of large-N expansion to derive non-trivial phase structure and non-perturbative particle spectra in $D = 3$ gauge theories with fermions, including supersymmetric nonlinear sigma-models, can be found in refs.[7,6] and references therein. For the role of large-N vector QFT models in the context of anti-de-Sitter/conformal-field-theory dualities in modern non-perturbative string theory, see ref.[8].

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BPHZL Renormalization

The mass-independent ("soft-mass") renormalization scheme of of Zimmermann-Lowenstein [1,2] is based on the standard BPHZ (Bogoliubov-Parasiuk-Hepp-Zimmermann) momentum-space subtraction procedure (the former is called *BPHZL renormalization* scheme for short). The general idea is to perform all subtractions in the integrands of ultraviolet (UV) divergent Feynman diagrams at zero external momenta and at *zero values of the mass parameters* except for those which by naive power counting would give rise to infra-red (IR) divergences, so that the latter subtractions are performed at zero external momenta but at *non-zero values of the mass parameters.*

Technically, this is accomplished in the following way:

(a) One rescales temporarily all dimensionfull (mass) parameters M entering the propagators and vertices of a diagram $M \to s^{d_M} M$ where d_M is the canonical mass dimension of M and at the end of the subtraction procedure the auxiliary parameter s is set to $s = 1$.

(b) For the masses in the propagators of the fundamental bosonic (φ) and fermionic (ψ) matter fields one assigns temporarily a slightly more complex dependence on the auxiliary parameter s:

$$-i\Big[(sm_\varphi + (1-s)\mu)^2 + P^2(p,k)\Big]^{-1} \quad , \quad -i\,(sm_\psi - \gamma^\nu P_\nu(p,k))\,\Big[(sm_\psi + (1-s)\mu)^2 + P^2(p,k)\Big]^{-1}$$

where $P(p,k)$ is a linear combination of external $\{p\}$ and internal $\{k\}$ momenta, μ is arbitrary renormalization mass scale and again at the end of the subtraction procedure one sets $s = 1$.

(c) The momentum space Taylor subtraction operators $\tau^{\delta(\Gamma),\rho(\Gamma)}$ in the fundamental "forest formula" of the recurrsive BPHZ subtraction scheme, acting on the integrand of a UV-divergent (sub)diagram Γ, are now defined as: $1-\tau^{\delta(\Gamma),\rho(\Gamma)} = \left(1 - t^{\rho(\Gamma)-1}_{\{p\},s-1}\right)\left(1 - t^{\delta(\Gamma)}_{\{p\},s}\right)$. Here $\delta(\Gamma)$ and $\rho(\Gamma)$ are the UV and IR indices of the (sub)diagram Γ determined from the asymptotic behaviour of its integrand for large internal momenta, and for small internal momenta at vanishing external momenta and all masses set to zero, respectively. $t^n_{x,y}$ denotes the usual Taylor subtraction operator: $t^n_{x,y}F(x,y) \equiv \sum_{k,l=0\,,\,k+l\leq n}^{n} \frac{x^k}{k!}\frac{y^l}{l!}\frac{\partial^{k+l}F}{\partial x^k \partial y^l}\Big|_{x=0,y=0}$.

BPHZL renormalization has found a non-trivial application in the systematic renormalization of non-perturbative large-N expansions of nonlinear sigma-models and their supersymmetric extensions [3,4] which look "non-renormalizable" from the point of view of naive perturbation theory w.r.t. coupling constants and which display rich phase structure with various "low-temperature" phases containing massless particles.

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